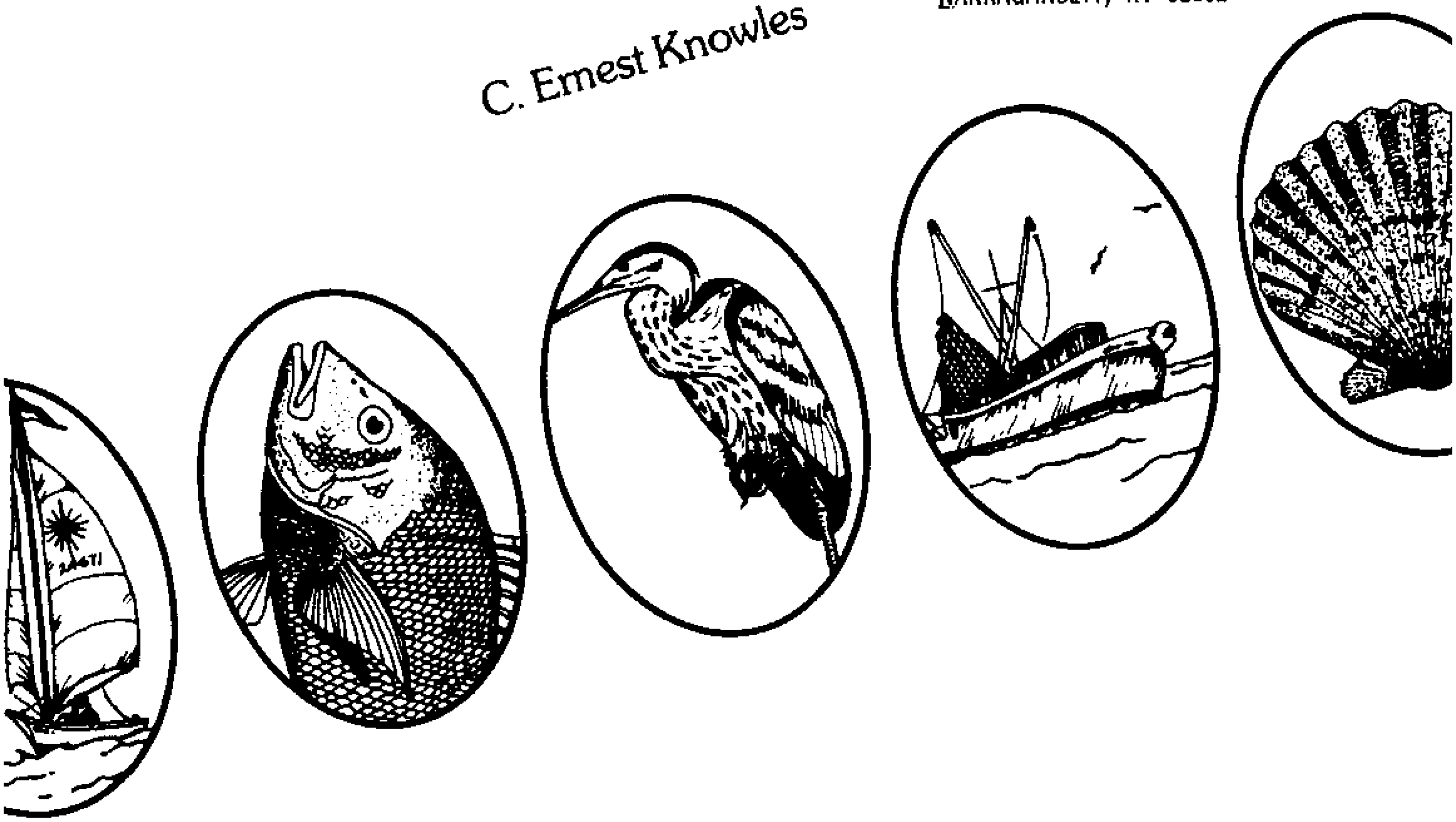
Working Paper 81-6

Estimation of Surface Gravity Waves From Subsurface Pressure Records For Estuarine Basins

C. Ernest Knowles



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ESTIMATION OF SURFACE GRAVITY WAVES FROM SUBSURFACE PRESSURE RECORDS FOR ESTUARINE BASINS

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This work was sponsored by the Office of Sea Grant, NOAA, U.S. Department of Commerce, under Grant No. NA79AA-D-00048 and the North Carolina Department of Administration.

Department of Marine, Earth and Atmospheric Sciences
Contribution No. 81-13

UNC Sea Grant College Publication UNC-SG-WP-81-6

September, 1981

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ABSTRACT

According to small-amplitude theory, the surface gravity-wave energy spectrum can be estimated from a subsurface pressure-fluctuation spectrum by applying a factor K that compensates for the attenuation of surface-wave amplitude as the depth below the water surface and the wave frequency increase.

These are a number of linear and nonlinear environmental factors, however, that cause K to be invalid over most of the spectrums frequency range. Numerous attempts have been made to emipirically derive a valid correction factor n that could be applied to K to give a better estimate of the surface spectrum. This report discusses some of the reasons why the empirical factors vary so greatly and recommends Graces' (1978) equation for n as a function of the non-dimensional frequency parameter kh (where $k=2\pi/L$ is the local wavenumber, h the local depth and L the wavelength) specifically for use in an estuarine environment.

The report also evaluates the maximum limit (K_m) on the magnitude of K suggested by Esteva and Harris (1970); where relative depth d/h (d is the pressure-transducer height above the bottom) and k_oh (k_o is the deep-water wavenumber) were the independent variables. The choice of K_m may be made unimportant if d is selected beforehand, using an equation (derived as a part of the present study and included in the appendix) for the minimum d/h limit affected by the choice of K_m.

Finally, the appendix also includes the equations and a FORTRAN IV SUBROUTINE program to estimate kh from k_oh. Several forms for the initial estimate of kh (i.e. $(kh)_o$) were evaluated; convergence to within a tolerance <0.0001 over the whole range of k_oh was obtained for $N\leq 8$ (where N is the number of iterations) and for $3\leq N\leq 5$ in the range of $2.55\leq k_oh<3.25$ using $(kh)_o=k_oh/\tanh(k_oh)$.

Introduction

There are two basic ways of measuring surface gravity-waves in coastal and estuarine waters; one uses a surface-wave profiler (e.g., continuous-wire or step-resistance gauges) that produces a time-series record that should represent the actual surface-wave profile, while the other (a sensitive, subsurface pressure transducer) produces a time series of pressure fluctuations proportional to the elevation of the surface waves. Relating the two records is not without difficulty, however, because the amplitude of the surface fluctuations attenuates as the depth of water below the surface and the wave frequency increases. The surface-wave profile in theory can be estimated from the pressure record by applying an attenuation-compensation factor, but this factor never is known precisely; and even if a reasonably good factor can be applied, much of the high frequency information will be lost because of instrument response limitations to very small pressure fluctuations.

According to linear, small-amplitude theory the surface-wave energy spectrum, $S(f)$, can be estimated from the pressure-fluctuation spectrum, $S_p(f)$, by applying the compensation factor

$$K = \cosh kh/\cosh kd \qquad (1)$$

to $S_p(f)$, i.e.

$$S(f) = K^2 S_p(f), \qquad (2)$$

where h is the local water depth, d is the pressure transducer height above the bottom and k is the local wavenumber ($=2\pi/L$, where L is the local wavelength). The parameter kh is related to and can be estimated from wave frequency f by finding the root of the linear-theory dispersion equation,

$$\frac{(2\pi f)^2 h}{g} = kh \tanh kh, \qquad (3)$$

using a numerical iterative scheme (see Appendix A). The deep-water (i.e. where $h > L/2$) frequency parameter $k_o h$ is estimated from (3) for large kh (as $\tanh kh \to 1$) by

$$k_o h = \frac{(2\pi f)^2 h}{g}, \qquad (4)$$

where k_o is the deep-water wavenumber.

There are a number of environmental and other linear and nonlinear factors that cause (1) to be invalid over most of the range of kh. Any factor that alters the frequency or shape of the waves will affect (1), (2) or (3). For instance:

(a) Implicit in (2) is the assumption that gravity-waves are distributed according to the zero-mean Gaussian model, yet it is known that wave profiles are not sinusodial.

(b) Waves recorded by pressure transducers in the presence of a current (particularly where the current sets directly against or with the waves) also will give erroneous estimates of K if (3) is used to obtain kh. In the presence of a current, (3) should be replaced by

$$(\omega-\underset{\sim}{k}\cdot\underset{\sim}{U})^2 \frac{h}{g} = (kh)_c \tanh(kh)_c, \qquad (5)$$

where ω ($=2\pi f_c$) is the apparent frequency (that actually observed at a fixed point) and $\underset{\sim}{k}\cdot\underset{\sim}{U}$ the advective frequency (due to the presence of a current with velocity vector $\underset{\sim}{U}$); both are related to the intrinsic frequency σ ($=2\pi f$, which is the frequency observed if traveling in a reference frame fixed to the current by

$$\omega = \sigma+\underset{\sim}{k}\cdot\underset{\sim}{U}. \qquad (6)$$

Note that the kh estimated from (3) and (5) may be (if $|\underset{\sim}{U}|$ or $|\underset{\sim}{k}|$ are large) quite different; Table 1 (from Peregrine, 1976; p. 25) shows the minimum period of waves as a function of depth in meters for which an adverse current of 50 cm/sec ($\approx$ 1 knot and easily obtained in tidal inlets) may be ignored in estimating (1) and (2) if errors are to be less than five and 20%, respectively. When $|\underset{\sim}{U}|$ is small and/or $\underset{\sim}{U}$ is nearly perpendicular to $\underset{\sim}{k}$, $\underset{\sim}{k}\cdot\underset{\sim}{U}\to 0$, (5) approaches (3) and the periods shown in Table 1 tend toward zero.

Table 1. Minimum period of a wave for which an adverse current of 50 cm/sec may be ignored in calculating surface amplitudes from bottom-pressure measurements if errors are to be less than five and 20%

	Depth (m)		
	1	2	5
Period (sec) with error of 5%	4.5	5.4	6.9
Period (sec) with error of 20%	2.7	3.2	4.3

(c) Finally (but by no means exclusively), the shape, composition and slope of the bottom can affect the high wavenumber portion of the pressure field being recorded by the transducer; any perturbation near the tranducer can introduce second-order nonlinear pressure fluctuations.

In an attempt to compensate for some of the factors that cause (1) to be an invalid estimator of K, investigators have made a number of simultaneous surface profile and subsurface-pressure fluctuation measurements and derived an empirical attenuation correction factor, n, which modifies K and which can be applied to (2), i.e.

$$S(f) = n^2 K^2 S_p(f), \qquad (7)$$

to give a better estimate of the surface-wave energy spectrum. The remainder of this report will discuss the deviation of these empirical factors and some of the reasons why they vary so greatly, recommend an n for use in shallow estuarine basins (where waves have periods generally less than four secs), and discuss the use of a maximum K value that is applicable over a wide range of frequencies.

Wave-Pressure Empirical Correction Factors

Empirical correction factors have been derived from simultaneous surface- and subsurface-pressure fluctuation measurements obtained both from controlled, nearly monochromatic-wave laboratory wave tank experiments and from irregular-wave ocean field experiments. In general, the field measurements have been in deeper water and for wave periods much greater than encountered in estuaries; some of the wave tank experiments, while including wave periods in the estuarine range, could be subject to wave- and depth-scaling problems.

Two basic approaches for deriving n from the simultaneously recorded data also have been used; both compare the surface measurements with the corrected pressure measurements. The deterministic approach makes a wave-to-wave comparison of the time series. The data obtained by most early researchers (Folsom, 1947, 1949; Seiwell, 1948; Gerhardt et al, 1955) generally were so scattered that only a mean value for n was provided (i.e. no functional relationship between n and kh (or f) could be found -see Grace 1970 for a summary of these experiments). More recent deterministic approaches (Grace, 1978) have been more successful. The other approach considers the waves to be the result of Gaussian random processes and makes comparisons of the surface and compensated-pressure spectra (Hom-ma et al, 1966; Esteva and Harris, 1970).

These later deterministic and spectral approaches have yielded n values that are a function of kh (or f) and that tend to decrease below unity as kh increases. Many still show, however, a wide discrepancy in the n relationships established from different sets of data. Much of the discrepancy might be attributable to field versus wave tank scaling, but even from two outwardly similar field experiments the data have yielded different n.

Undoubtedly, some of these differences could have been explained by including currents in the determination of kh, but no field investigator who made these simultaneous surface/pressure measurements measured currents at the site. Peregrine (1976) demonstrated, however, that much of the scatter in Draper's (1967) n measurements could be caused by a 1.2 m/sec current in six meters of water setting against the waves at the sensor site. Not all field investigators included details on the bottom roughness either, so much of the scatter at higher f may be the result of bottom perturbations.

One other possible cause for the scatter in the higher frequency portion of the data (and the associated discrepancies in the functional n relationship derived from it) may be related to the relative depth, d/h. Other investigators use z/h as relative depth, where z is the transducer depth below the surface rather than d, the distance above the bottom. This author prefers d/h for hand calculation of K and this definition will be used in this report; and because z/h=1-d/h, one value can be obtained easily from the other.

As can be seen from (1), K = K(k,h,d); n was shown to be a function of k and h, so should there be also a dependence of n on d? Linear theory suggests there should be, at least for values of relative depth $d/h < 0.2$. The functional dependence of K on these three parameters is shown in Figure 1, where K/K_a is plotted versus d/h for various values of k_oh, and K_a is the asymptotic value of K as $\cosh kd \to 1$, i.e.

$$K_a = \cosh(kh). \tag{8}$$

Note in particular that K changes much more rapidly as k_oh increases; K has come to within 10% of K_a for $k_o = 0.5$ at $d/h = 0.6$, but for $k_oh = 3.0$ (i.e. near the linear-theory deep-water wave limit) K does not come to within 10% until $d/h = 0.15$. This rapid change in K for larger values of k_oh (higher f) suggests a major difficulty in using pressure transducers in shallow estuaries and in making precise surface/pressure comparisons when $d/h > 0.2$; that part of the correction factor K most subject to rapid change amplifies that part of the spectrum that is likely to have the most measurement error.

If small-amplitude theory suggests some functional dependence of K on d/h, do the comparison experiments show the same functional dependence of n on d/h? The wave tank experiments (Folsom, 1947, Hom-ma et al., 1966) showed no apparent correlation. Most of those conducting the field

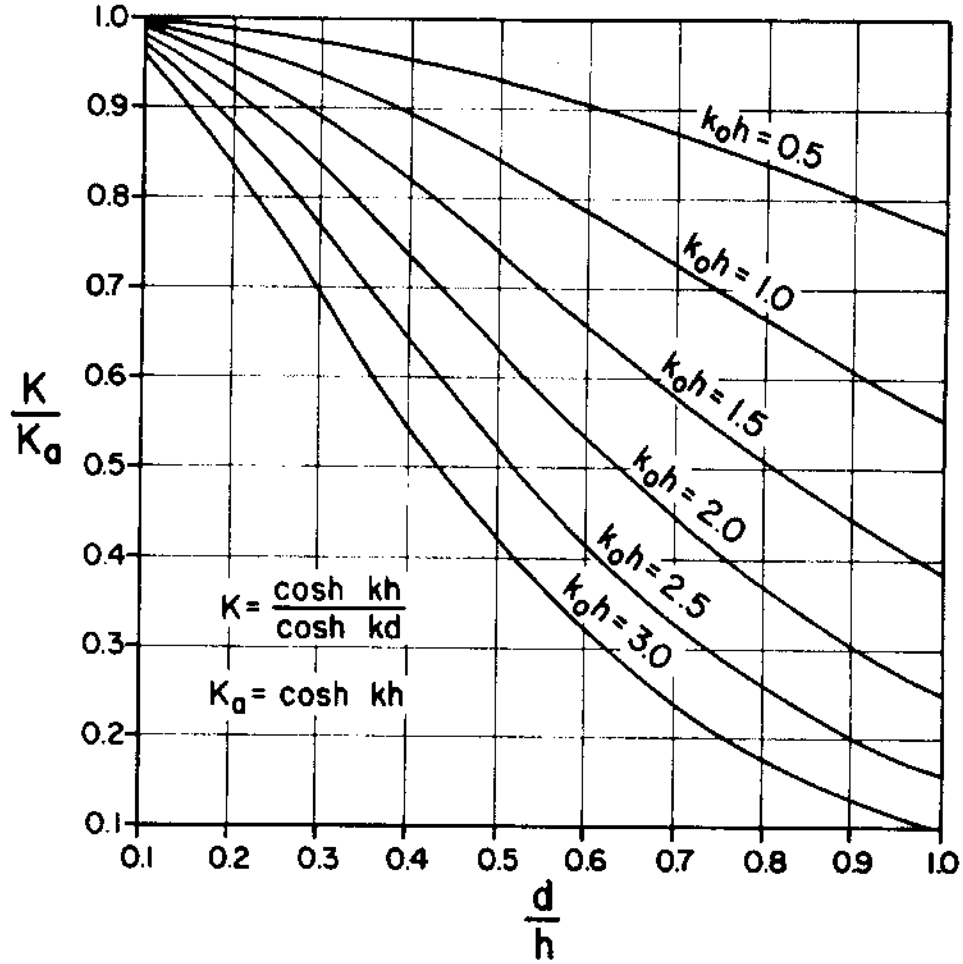

Figure 1. Plot of K/Ka as a function of h/d for several values of k_oh, where h is local water depth, d transducer height above the bottom, K the linear-theory pressure attenuation compensation coefficient and Ka the asymptotic value of K as $kd \to o$.

experiments either mounted their instruments on the bottom ($z/h \to 1$ or $d/h \to o$), in which case $K \to K_a$, or they did not report their transducer height, or they made no note of differences in n due to d/h. All their data usually were included in the analysis, however, and should show the most scatter in n when $d/h > 0.2$ and the least when $d/h < 0.2$; an examination of these data seems to suggest that this is the case. The least scatter in n values (and least mean departure of n from 1 for small $k_o h$) occurs for those data (Esteva and Harris, 1970-lower gauge; Grace, 1978-lab) where $z/h > 0.93$ or $d/h < 0.07$ (and K for all values of $k_o h$ are within 5% of K_a).

In conclusion, it is clear that the linear-theory attenuation compensation factor K must be modified by an empirical correction factor n to provide an adequate estimate of the surface wave spectra. The linear and nonlinear conditions discussed above probably have contributed to the scatter in the data and the wide discrepancy in the n relationships reported in the literature; the uncertainty caused by these different results makes choosing the appropriate n relationship, especially for use in an estuarine environment, difficult.

Because the vast majority of field measurements have been made in water deeper than and with wave periods greater than those found in estuaries and until controlled simultaneous measurements are made there, it seems prudent to look to laboratory wave tank experiments for the proper n. In order to avoid the unique scaling limitations found in some wave tanks (in which Phillips, 1977, argues that the balance of dynamical processes are different than in the field), it is suggested that Grace's (1978) field and wave tank-derived equation for n as a function of kh be used:

$$n(kh) = \begin{cases} 1.550 - 4.50kh/2\pi; & \dfrac{kh}{2\pi} < 0.1 \\ 1.175 - 0.75kh/2\pi; & 0.1 < \dfrac{kh}{2\pi} < 0.5. \end{cases} \tag{9}$$

The first equation comes from Grace's analysis of ocean field data and the second from his controlled monochromatic-wave experiment in Oregon State's Wave Research Facility, where the wavetank's large dimensions (104.2m long, 3.66 m wide and 4.57 m deep) provide conditions approaching those found in the ocean. The measurements were made for $d/h < 0.05$ so relative depth should not be a factor in affecting K; and because they were made in a wave tank, there will be no current effects and probably no bottom effects to contaminate the determination of kh and n. As shown

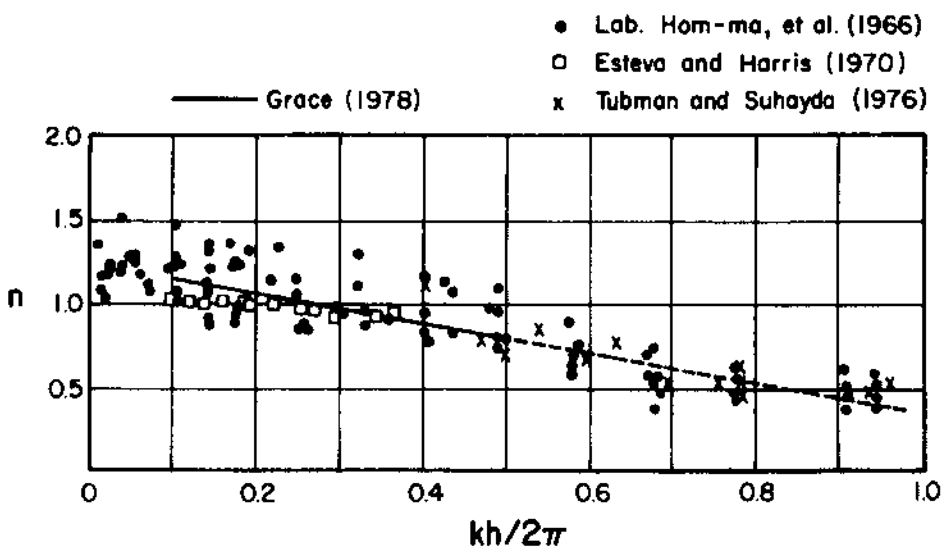

Figure 2. Empirical correction factors n(kh) as a function of $kh/2\pi$. Solid line is from equation (9), for Grace's (1978) laboratory data; the dashed line extends beyond $kh/2\pi = 0.5$, which was his data's upper limit. The other data shown includes Hom-ma et al's (1966) laboratory, Esteva and Harris' (1970) lower wave-gauge field and Tubman and Suhayda's (1976) field data.

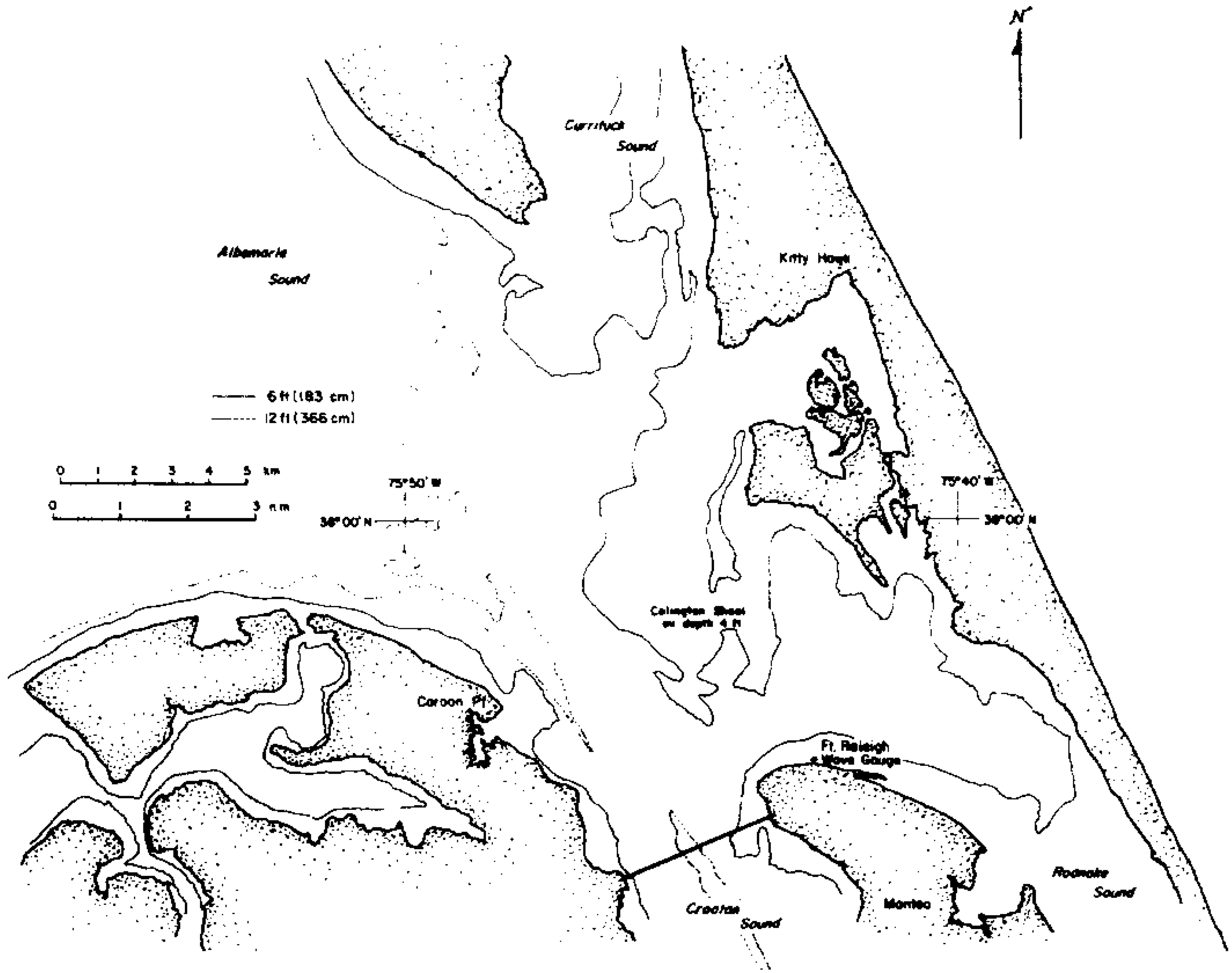

Figure 3. Topographic and bathymetric chart of SE Albemarle basin adjacent to Fort Raleigh wave-gauge site.

in Figure 2, Hom-ma et al's (1966) wave tank data (where scatter may be due to smaller tank dimensions), Esteva and Harris (1970) lower-gauge field data (converted to be a function of kh) and Tubman and Suhayda's (1976) field data seem to fit the second equation in (9) for $kh/2\pi > 0.1$. Grace's equation has been extended beyond $kh/2\pi = 0.5$ (as shown by the dashed line), which seems to fit the other data as well.

The choice of this or any functional relation for n may help compensate for the inadequacies of using the linear theory K, but it must be emphasized that n does not account for currents present at the site; they should be evaluated separately and included when necessary in the determination of kh using (5).

Evaluation of a maximum K Limit

Even if K could be calculated with confidence (and modified by a valid empirical n), should it have an absolute maximum value K_m (and, therefore, a maximum frequency cut-off)? Esteva and Harris (1970) recommended for their ocean installation and wave conditions that a K_m of not more than five[1] (which corresponds to a wave-induced pressure at the transducer that is 20% of the surface value) be used. For $d/h \to 0$ (as $K \to K_a$), this corresponds to a maximum deep-water frequency parameter $(k_oh)_a = 2.3$.

While this reduction limit to 20% of the surface value usually only affects the higher frequency portion of the wave spectrum, it could be argued that the limit is too severe for estuarine depths and wave frequencies; this author's experience with wave data at the Ft. Raleigh wave site (c.f. Figure 3) suggests that a 15% limit may be just as appropriate (i.e., $K_m = 6.7$ and $(k_oh)_a = 2.6$).

The parameter k_oh (because it is more easily calculated than, and is uniquely related to kh by (3)) has been plotted versus d/h for various values of K (c.f. Figure 4); some general observations can be made from the figure. Ignoring for the moment the area outlined by heavy lines (which will be discussed below) note that the 15% and 20% limit lines cross $k_oh = \pi$ (near the small-amplitude deep-water wave boundary) at $d/h = 0.37$ and $d/h = 0.48$, respectively. These d/h values would then constitute the maximum d/h limit for the K_m chosen; for all d/h greater than these limits $K < K_m$.

The minimum d/h value affected by any choice of K_m can be obtained from

---

[1] Esteva and Harris (1970) defined (2) in the form $S(f) = K_{eh}S_p(f)$; i.e., the K used in this report in (2) differs from theirs as $K = \sqrt{K_{eh}}$, and their suggested limit, $(K_{eh})_m = 25$, is for this study $K_m = 5$.

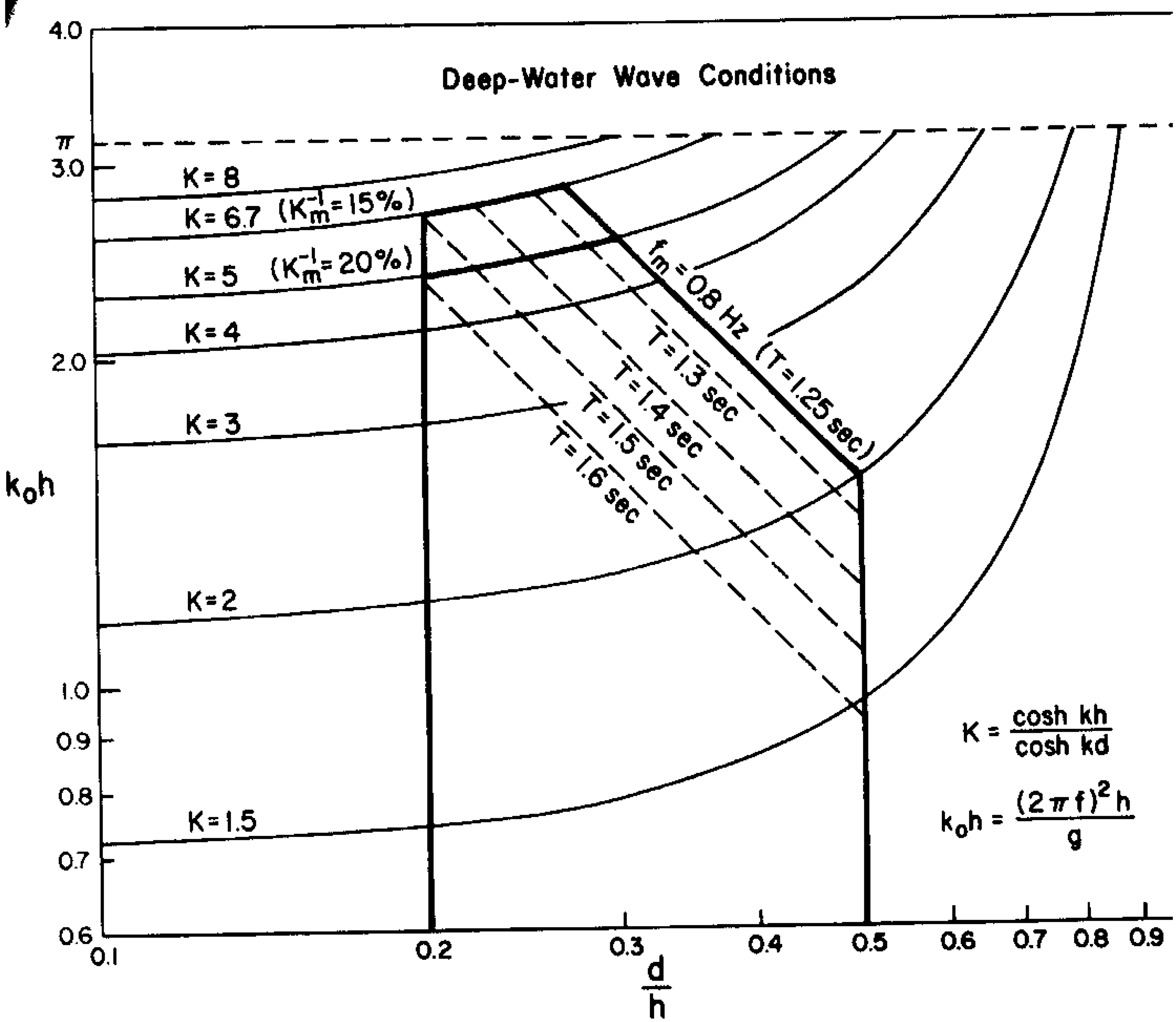

Figure 4. Plot of K versus h/d for various values of k_oh. The "limiting-window" (outlined by heavy lines) and the dashed lines contained therein, are defined from and represent dimensional data characteristic of estuarine waves found at Fort Raleigh wave site.

$$\left.\frac{d}{h}\right|_{min} = (kh)^{-1} \ln (\gamma + \sqrt{\gamma^2 - 1}) \qquad (10)$$

(see Appendix B for its derivation), where

$$\gamma = \frac{\cosh kh}{K_m} > 1 \qquad (11)$$

and $K<K_m$. Equation (10) also can be used to select a transducer height d, for instance, so that $K<K_m$ if h and the minimum wave period (of interest in a particular environment) are known. For example, typical estuarine values of h and T_{min} might be 150 cm and 1.6 sec, respectively; for $K_m=5$ (20% limit), (10) would give a $(d/h)_{min} = 0.2$, or d=30. Therefore, $K<K_m$ for d> 30 cm.

For d/h>0.2, K may be more restricted by the high frequency limitations of the pressure transducer ($f_m < 0.8H_z$ for most sensors) than by K_m. To emphasize the impact that this frequency limitation and the other limitations discussed above might have in an estuarine environment, and to demonstrate the differences between the 15% and 20% limits on K for the higher frequency (lower period) portion of the wave spectrum, additional dimensional data from the Ft. Raleigh wave site have been included in Figure 4 to define a "limiting window" (outlined by heavy lines). The two top curved-lines of the window are the 15% ($K_m = 6.7$) and the 20% ($K_m = 5.0$) limits on K, respectively. The two vertical lines are the d/h limits 0.2<d/h<0.5 imposed by the Ft. Raleigh depth range extremes (60 cm <h<150cm) and transducer height (d=30cm), and the sloping right boundary is the limit on K associated with the transducer high-frequency cut-off ($f_m=0.8H_z$) as a function of d/h.

Some general observations for the Ft. Raleigh data also can be made from Figure 4. Wave periods T > 1.6 sec (f<0.625HZ) will never have K's that exceed K_m, so the choice of K_m is irrelevant; when T<1.6 sec, however, the 20% limit can be much more restrictive than the 15% limit. For instance, at d/h=0.25 the 20% limit will be invoked for T < 1.4 secs, but the 15% limit will not be until T < 1.3 sec. Note also the different d/h ranges for the Fort Raleigh data that will be subject to the transducer high-frequency cut-off (f_m) as a result of these two K_m limits; for the 20% limit, f_m is more restrictive than K_m for just over three-fourths of the d/h range (d/h>0.27), but for the 15% limit f_m is more restrictive than K_m for about two-thirds of the d/h range (d/h>0.3).

In conclusion, these observations from Figure 4 reemphasize the fact that for most estuarine water depths, the imposition of a K_m is of concern only for the higher frequency portion of the wave spectrum; and for d/h>0.27, K in general is more restricted by f_m than K_m. Therefore, even though the data for Ft. Raleigh suggests that the 20% limitation may be too restrictive for estuarine conditions, the limit chosen probably should depend more upon the investigator's confidence in the pressure sensor used than in an "exact" percent reduction.

Conclusions and Recommendations for Further Study

Grace's (1978) equation for the empirical correction factor n given by (9) seems to be the best choice for use in an estuarine environment. It would seem worthwhile, however, to conduct a surface-wave profile, subsurface-pressure fluctuation field experiment in an estuarine environment within the relative depth range $0.2<d/h<0.5$ to verify this choice for n. The experiment also should include a careful monitoring of water depth h, transducer height d, the bottom roughness and the presence of currents at the transducer depth and location (so that corrections to kh may be made).

The imposition of a maximum limit on K (K_m) also is probably justified. The choice of a 15% or 20% limit, however, probably will be important only for the higher frequency portion of the wave spectrum, and may be made unimportant if (10) is used beforehand to select d. Because the K_m and $(d/h)_{min}$ were derived from small-amplitude theory, it seems more logical to apply the K_m limitations to the calculation of K (using equation (1)) before K is modified by n. As can be seen in Figure 2, this would result, (for the range of kh where K is limited by K_m), in a value for the product nK that will always be less than K.

ACKNOWLEDGEMENTS

Special thanks to G.S. Janowitz for his helpful review of the manuscript and for suggesting the derivation of $(d/h)_{min}$.

REFERENCES

Draper, L., 1957. Attenuation of sea waves with depth. Hoinlle Blanche, 12, pp. 926-931.

Esteva, D. and D. L. Harris, 1970. Comparison of pressure and staff wave gauge records. Proc. 12th Coastal Eng. Conf., Washington, DC, pp. 101-116.

Folsom, R. G., 1947. Measurement of Ocean Waves, Trans. AGU, Vol. 28(b), pp. 875-881.

Gerhardt, J. R., K. H. Jehn, and I. Katz, 1955. A comparison of step-, pressure-, and continuous-wire-gauge wave recordings in the Golden Gate Channel. Trans. AGU, Vol. 36(2), pp. 235-250.

Grace, R. A., 1970. How to measure waves, Ocean Ind., 9(2), pp.65-69.

Grace, R. A., 1978. Surface wave heights from pressure records. Coastal Eng., Vol. 2, pp. 55-67.

Hom-ma, M., K. Horikawa and S. Komori, 1966. Response characteristics of underwater wave gauge. Proc. 10th Coastal Eng. Conf., Tokyo, Japan, pp. 99-114.

Peregrine, D. H., 1976. Interaction of water waves and currents. Adv. Appl. Mech., 16, p. 9-117.

Phillips, O. M., 1977. The dynamics of the upper ocean, 2nd Ed., Cambridge Univ. Press, 336 p.

Seiwell, H. R., 1948. Results of research on surface waves of the Western North Atlantic. Papers in Phys. Ocean. and Met., MIT (WHOI), Vol. 10(4), August.

Tubman, M. W. and J. N. Suhayda, 1976. Wave action and bottom movements in fine sediments. Proc. 15th Coastal Eng. Conf., Honolulu, Hawaii, pp. 1168-1183.

APPENDIX A. Estimating local wavenumber given the deep water wavenumber and water depth.

A non-linear equation like (3) can be solved by numerical iteration. Rapid convergence to an acceptable value of kh can be achieved using a scheme that employs a rewritten (3); i.e.

$$(kh)_{2n-1} = k_oh/\tanh(kh)_{2(n-1)}, \qquad \text{(A-1)}$$

to obtain the first iterative estimate. The second estimate then is obtained by averaging the two previous ones; i.e.

$$(kh)_{2n} = 1/2\left[(kh)_{2n-1} + (kh)_{2(n-1)}\right], \qquad \text{(A-2)}$$

where $n = 1,2,3,\ldots,N$ and $(kh)_o$ (at $n = 1$) is an initial estimate of kh. If $(kh)_o$ is assumed equal to k_oh, this scheme will provide convergence to within a tolerance of < 0.0001 for $N < 10$ (i.e. iterating through (A-1) and (A-2) 10 times for 20 estimates). Convergence to within the same tolerance can be obtained for $N < 8$ by using as the initial estimate

$$(kh)_o = k_oh/\tanh\ (k_oh); \qquad \text{(A-3)}$$

in fact, convergence is achieved for $N = 2$ for $k_oh > 3.25$ and for $3 < N < 5$ in the range $2.55 < k_oh < 3.25$. Finally, a slight improvement in convergence ($N < 5$) is obtained for $0 < k_oh < 0.5$ by using

$$(kh)_o = k_oh/\tanh^{1/2}(k_oh) \qquad \text{(A-4)}$$

as the initial estimate.

Table A-1 is a FORTRAN IV subroutine that incorporates (A-1) and (A-2); double precision is used to reduce round-off. Z is the initial estimate $(kh)_o$ that can be simply equal to k_oh or obtained from (A-3) or (A-4); (A-3) is recommended as the best overall estimator of $(kh)_o$.

Table A 1. Iterative FORTRAN IV SUBROUTINE for finding root of transcendential equation of $k_o h$ = kh tanh kh

```
c
      SUBROUTINE WNUMB (DWWN, WNL, J, Z)
c
      REAL*8  DIFF, TOL, DUMMY, WN, WN, Z, DWWN, DUMMY I
c      TOL    TOLERABLE ERROR FOR SOLUTION CONVERGENCE
c      DWWN   DEEP WATER WAVENUMBER *DEPTH (koh)
c      WNL    LOCAL WAVENUMBER *DEPTH (kh)
c      Z      FIRST ESTIMATE TO START ITERATION (kh)o
c      J      COUNTER FOR # OF ITERATIONS (1/2 no. of estimates)
c
       J = 1
       DUMMY = Z
       TOL = 0.0001
c
   10  WN = DUMMY
       DUMMY1 = DWWN/DTANH(WN)
       DUMMY = (DUMMY1 + WN)/2.
       DIFF = DABS (DUMMY1 - DUMMY)
       J = J + 1
c
       IF (DIFF.GT.TOL) GO TO 10
  100  WNL = DUMMY
c
       RETURN
       END
```

APPENDIX B. Derivation of minimum d/h limit for a given K_m

Equation (1) can be written

$$K = \frac{\cosh kh}{\cosh (kh.\frac{d}{h})} \qquad (B-1)$$

A. If $K>K_m$, the application of K^2 in (2) is truncated, no matter the value of d/h.

B. If $K<K_m$, however, there are conditions where d/h will have a minimum value associated with the choice of K_m; i.e. when $K<K_m$

$$\frac{\cosh kh}{\cosh kd} < K_m \ . \qquad (B-2)$$

(1) If $\cosh kh<K_m$, then $K<K_m$ for all d/h (as can be seen from (B-1) and (B-2)), and d/h has no minimum value.

(2) If, on the other hand;

$$\cosh kh > K_m, \qquad (B-3)$$

then from (B-2)

$$\cosh kd > \frac{\cosh kh}{K_m} = \gamma \qquad (B-4)$$

where

$$\gamma>1, \qquad (B-5)$$

which, of course, satisfies (B-3).
From (B-4)

$$e^{kd} + e^{-kd} > 2\gamma, \qquad (B-6)$$

or in polynomial form (multiply (B-6) by e^{kd})

$$P(kd,\gamma) = e^{2kd} - 2\gamma e^{kd} + 1 > 0. \qquad \text{(B-7)}$$

At $kd = 0$, $e^{kd} = 1$ and $P = 2(1-\gamma)$; P increases from a minimum of $2(1-\gamma)$ with increasing kd, as shown in Figure B-1. If we let $e^{kd}=e^{kd_o}$ when $P=0$, then to satisfy (B-7), $e^{kd} > e^{kd_o}$ and

$$kd > kd_o, \qquad \text{(B-8)}$$

where kd_o is the minimum kd affected by the choice of Km. Equation (B-7) can be solved for e^{kd_o} using the binomial equation; i.e.

$$e^{kd_o} = \gamma \pm \sqrt{\gamma^2-1}\ , \qquad \text{(B-9)}$$

or

$$kd_o = \ln\ (\gamma + \sqrt{\gamma^2-1}), \qquad \text{(B-10)}$$

where the positive root is chosen to satisfy (B-8). The minimum d/h limit affected by K_m is, therefore,

$$\left.\frac{d}{h}\right|_{min} = (kh)^{-1}\ \ln(\gamma + \sqrt{\gamma^2-1}), \qquad \text{(B-11)}$$

for conditions $K < K_m, \gamma > 1$

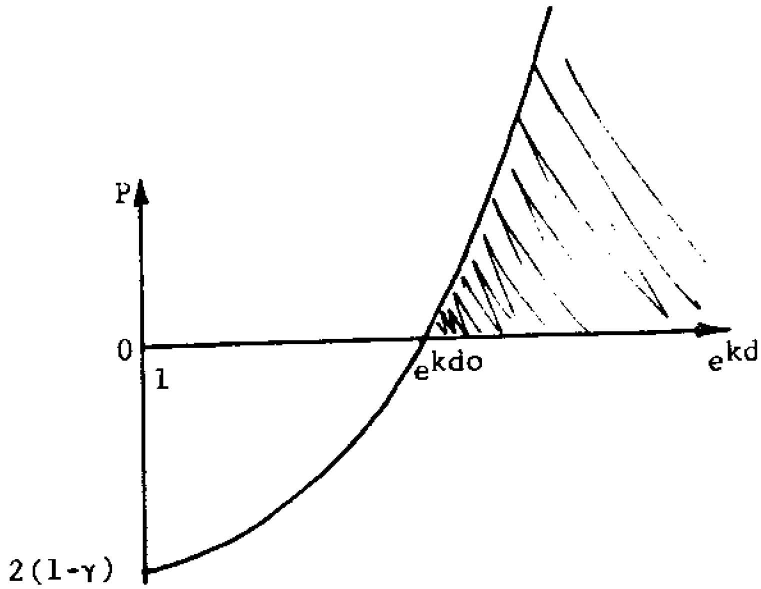

Figure B-1. Schematic of polynomial function P versus e^{kd} and γ. The shaded area indicates region where d/h may be limited by choice of K_m.